Combined Laminar Free- and Forced-convection Heat Transfer in External Flows

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A method is presented for calculating the shear stress and the rate of heat transfer in external flows for combined laminar forced and free convection. The parameter Gr/Re^2 is of fundamental importance in such problems. Numerical results are reported for the heating and cooling of upward flow past a vertical flat plate for three Prandtl numbers. It is found that the transition from forced to free convection is gradual, especially at high Prandtl numbers. The influence of free convection on the separation point is also examined.

It is well known that forced and free convection play a predominant role in determining the rate of heat transfer from a surface to a fluid moving past it. To date, however, the theoretical and experimental studies on this subject have been restricted, with few exceptions, to cases where either, but not both, of the two mechanisms is taken into account. These investigations have been very successful, especially in regions where the flow is laminar, and have resulted in experimentally verified theoretical predictions for the rate of heat transfer in forced or in free convection.

In general, however, heat is transferred by both mechanisms acting simultaneously. It is therefore of some interest and importance to be able to predict how the rate of heat transfer is affected by the combined action of both forced and free convection and to know under what conditions it is permissible to neglect one mode of transfer for the other.

A few studies have been made in this direction. Experimental rates of heat transfer for air in turbulent forced and free convection in tubes have been reported by Eckert and Diaguila (*1*). Van der Hegge Zijnen (*12*) has proposed an empirical correlation, substantiated by his own experimental data, for predicting the rate of heat transfer from cylinders to air under conditions of combined natural and forced convection. Ostrach (*7*), Hallman (*3*), and Hanratty (*4*) have investigated the velocity and temperature distribution in vertical pipes and channels with low Reynolds numbers, where free-convection effects must be included.

This paper will present a theoretical treatment of combined laminar free- and forced-convection heat transfer in external flows. The general discussion will deal first with laminar–boundary-layer flows past an arbitrary two-dimensional surface; then the simple but representative flat-plate problem will be studied in detail.

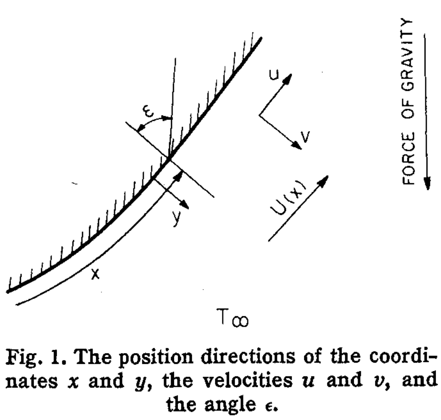

Fig. 1. The position directions of the coordinates x and y, the velocities u and v, and the angle ϵ.

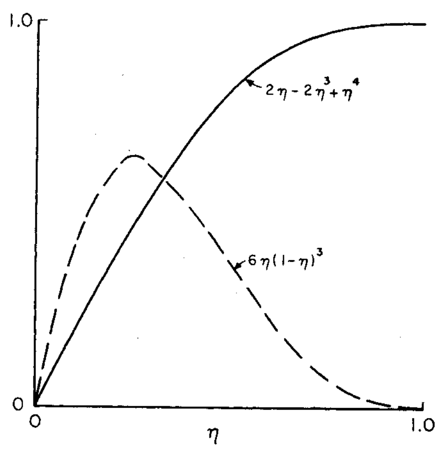

Fig. 2. The shapes of the two velocity profiles.

BASIC EQUATIONS AND THEIR INSPECTIONAL ANALYSIS

The laminar motion of a fluid at a temperature T_∞ past the arbitrary surface shown in Figure 1 will be considered. The surface temperature has the constant value T_w. It is well known that the basic equations which describe the behavior of this system, Navier-Stokes and energy, can be considerably simplified in many important cases by the use of Prandtl's boundary-layer theory. The resulting *boundary-layer equations* are generally used for such problems and can be found in the standard references (*2, 9, 10*). It is much more convenient, however, to start with the dimensionless form of the boundary-layer equations,

$$\left.\begin{aligned} u_1\frac{\partial u_1}{\partial x_1} + v_1\frac{\partial u_1}{\partial y_1} &= \pm\frac{Gr}{Re^2}\,\theta\sin\epsilon + U\frac{dU}{dx_1} + \frac{\partial^2 u_1}{\partial y_1{}^2} \\ \frac{\partial u_1}{\partial x_1} + \frac{\partial v_1}{\partial y_1} &= 0 \\ u_1\frac{\partial\theta}{\partial x_1} + v_1\frac{\partial\theta}{\partial y_1} &= \frac{1}{Pr}\frac{\partial^2\theta}{\partial y_1{}^2} \end{aligned}\right\} \quad (1)$$

with

$$\left.\begin{aligned} u_1 &= \frac{u}{U_\infty}, \quad v_1 = \frac{v}{U_\infty}\sqrt{Re}, \\ x_1 &= \frac{x}{L}, \quad y_1 = \frac{y}{L}\sqrt{Re} \\ \theta &= \frac{T - T_\infty}{T_w - T_\infty}, \\ \beta_1 &= g\beta\,|T_w - T_\infty| \end{aligned}\right\} \quad (2)$$

$Re = U_\infty L/\nu$ (the Reynolds number)

$Gr = \beta_1 L^3/\nu^2$ (the Grashof number)

$Pr = c_p\mu/k$ (the Prandtl number)

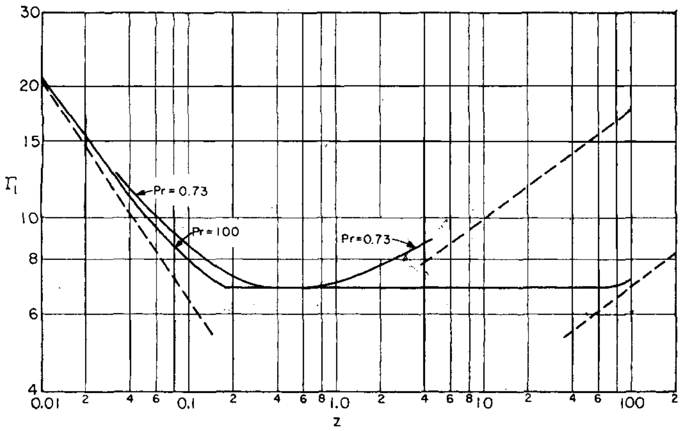

Fig. 3. Γ_1 heating vs. z.

β is the expansion coefficient of the fluid and is defined by

$$\frac{\rho_\infty}{\rho} = 1 + \beta(T - T_\infty) \qquad (3)$$

The boundary conditions are at

$$\left.\begin{array}{lll} x_1 = 0 & u_1 = U(0) & \theta = 0 \\ y_1 = 0 & u_1 = v_1 = 0 & \theta = 1 \\ y_1 = \infty & u_1 = U(x_1) & \theta = 0 \end{array}\right\} \qquad (4)$$

Although Equations (1) do not seem to have been given in this dimensionless form in the usual texts on the subject, they can readily be obtained from the boundary-layer equations by applying the transformations shown above. The buoyancy term $(Gr/Re^2)\theta \sin \epsilon$ is positive for the heating problem ($T_w > T_\infty$) and negative when $T_w < T_\infty$. It should be noted that terms usually of minor importance in low-speed flows—viscous dissipation and compression work—have been omitted from Equations (1), and that all the fluid properties except the density ρ are assumed constant.

For a given surface geometry the only parameters in the system of Equations (1) and (4) are the two dimensionless groups, Pr and Gr/Re^2. Therefore the expression for Nu becomes

$$Nu \equiv -L\left(\frac{\partial \theta}{\partial y}\right)_{y=0} = -\left(\frac{\partial \theta}{\partial y}\right)_{y_1=0} \cdot \sqrt{Re} = \sqrt{Re}\, \overset{(0)}{f_1}\left(Pr, x_1, \frac{Gr}{Re^2}\right)$$

where $\overset{(0)}{f_1}$ is a function of the indicated three variables only. However, in general one is interested in the mean Nusselt number for the surface, which is obtained from the foregoing expression by integrating with respect to x_1 from zero to a specified upper limit, which of course must not be past the separation point of the boundary layer, so that

$$Nu_m = \sqrt{Re}\, f_1\left(Pr, \frac{Gr}{Re^2}\right) \qquad (5)$$

Similarly one can show that $\overline{\tau_0}$ is given by

$$\frac{\overline{\tau_0}}{\rho U_\infty^2} \sqrt{Re} = f_2\left(Pr, \frac{Gr}{Re^2}\right) \qquad (6)$$

Naturally the functions f_1 and f_2, the form of which is determined by the surface geometry and the type of problem considered (heating or cooling), can be obtained only by solving Equations (1)

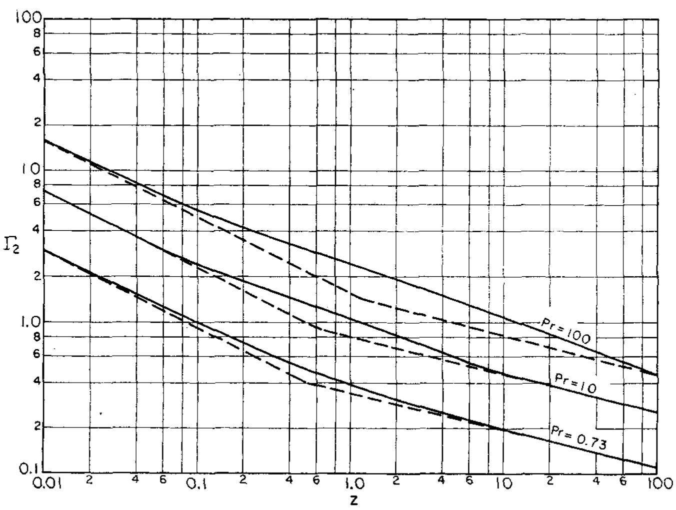

Fig. 5. Γ_2 heating vs. z.

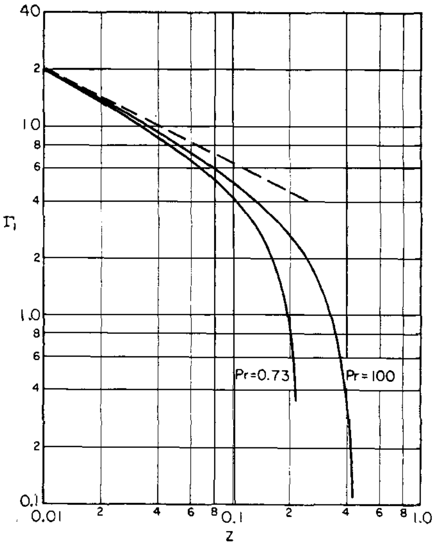

Fig. 4. Γ_1 cooling vs. z.

and (4), but the two relations above should prove useful in correlating and interpreting experimental data. It is also clear that, aside from the Prandtl number Pr, the parameter (Gr/Re^2) is of fundamental importance in the study of combined natural and forced-convection heat transfer. Finally it can easily be shown that, as expected, Equations (5) and (6) reduce to the corresponding equations for forced- or free-convection heat transfer respectively as the parameter (Gr/Re^2) approaches zero or infinity. What has still to be determined for each surface geometry and type of problem (heating or cooling) is

1. The form of the functions f_1 and f_2 in Equations (5) and (6) respectively and in particular the variation between their limiting forms for $(Gr/Re^2) \rightarrow 0$ and $(Gr/Re^2) \rightarrow \infty$ respectively.
2. The way in which the separation point, if it exists, is affected by the parameters Gr/Re^2 and Pr.

These two questions will be discussed in some detail in the next section for the special geometry of a vertical flat plate. The method to be presented for solving the problem is however general and can be used for more complicated geometrical surfaces.

COMBINED LAMINAR FREE- AND FORCED-CONVECTION HEAT TRANSFER FROM A VERTICAL FLAT PLATE

For the flow past a vertical flat plate, $U = 1$ and $\sin \epsilon = 1$, and so Equations (1) become

$$u_1 \frac{\partial u_1}{\partial z} + v_2 \frac{\partial u_1}{\partial y_2} = \pm\theta + \frac{\partial^2 u_1}{\partial y_2^2} \qquad (7a)$$

$$\frac{\partial u_1}{\partial z} + \frac{\partial v_2}{\partial y_2} = 0 \qquad (7b)$$

$$u_1 \frac{\partial \theta}{\partial z} + v_2 \frac{\partial \theta}{\partial y_2} = \frac{1}{Pr} \frac{\partial^2 \theta}{\partial y_2^2} \qquad (7c)$$

where

$$z = x_1 \frac{Gr}{Re^2},$$

$$y_2 = y_1\sqrt{\frac{Gr}{Re^2}}, \quad (8)$$

and

$$v_2 = v_1\sqrt{\frac{Re^2}{Gr}}$$

Furthermore the boundary conditions are At

$$z = 0 \quad u_1 = 1, \quad \theta = 0$$
$$y_2 = 0 \quad u_1 = v_2 = 0, \quad \theta = 1 \quad (9)$$
$$y_2 = \infty \quad u_1 = 1, \quad \theta = 0$$

Also since the flat plate has no characteristic length, it is simpler to set $L = x$ and to base both the Grashof and the Reynolds numbers on x, the distance from the leading edge of the plate. Thus

$$Gr = \frac{\beta_1 x^3}{\nu^2} \; Re = \frac{U_\infty x}{\nu}, \text{ and } z = \frac{Gr}{Re^2}$$

The system of Equations (7), (8), and (9), which contains the Prandtl number Pr as a single parameter, can be solved exactly by a numerical procedure only and with considerable difficulty. Therefore the approximate Pohlhausen–von Karman momentum integral method (*2, 10*) will be used. It is recognized that this method has a serious disadvantage common to most "approximate" treatments; it is not possible to estimate the accuracy and reliability of the answers derived from it. However, for all the simple problems having exact solutions of the boundary-layer equations, the integral method usually gives very satisfactory results, except for extreme cases. The two momentum integrals are

$$\frac{d}{dz}\int_0^\infty u(u-1)\,dy_2 = \pm\int_0^\infty \theta\,dy_2 - \left(\frac{\partial u}{\partial y_2}\right)_{y_2=0} \quad (10)$$

and

$$\frac{d}{dz}\int_0^\infty u\theta\,dy_2 = -\frac{1}{Pr}\left(\frac{d\theta}{dy_2}\right)_{y_2=0} \quad (11)$$

The next step consists of assuming likely velocity and temperature profiles, each containing one parameter which is a function of z. These profiles are usually chosen to be polynomials which satisfy the boundary conditions, Equation (9), and certain compatibility relations. The profiles selected for this problem are

$$u_1 = (2\eta - 2\eta^3 + \eta^4) \pm \frac{\delta^2\eta}{6}(1-\eta)^3 \text{ for } \eta \leq 1 \quad (12)$$

$$u_1 = 1 \text{ for } \eta \geq 1 \text{ where } \eta = \frac{y_2}{\delta}$$

and

$$\theta = 1 - \frac{3}{2}\eta_T + \frac{\eta_T^3}{2} \quad \text{for} \quad \eta_T = \leq 1$$

$$\theta = 0 \text{ for } \eta_T \geq 1$$

$$\text{where } \eta_T = \frac{y_2}{\delta_T} \quad (13)$$

δ and δ_T are both functions of z and must satisfy Equations (10) and (11) when the assumed profiles for u_1 and θ are substituted in them. It can readily be verified that these profiles satisfy Equation (9) and the following compatibility conditions:

$$\frac{\partial u_1}{\partial y_2} = \frac{\partial^2 u}{\partial y_2^2} = 0 \text{ at } y_2 = \delta \quad (14a)$$

$$\frac{\partial \theta}{\partial y_2} = 0 \text{ at } y_2 = \delta_T \quad (14b)$$

$$\frac{\partial^2 u}{\partial y_2^2} \pm 1 = 0 \text{ at } y_2 = 0 \quad (14c)$$

[Equation (7*a*) is satisfied at the surface.]

$$\frac{\partial^2 \theta}{\partial y_2^2} = 0 \text{ at } y_2 = 0 \quad (14d)$$

[Equation (7*c*) is satisfied at the surface.]

The actual selection of the profiles and the compatibility relations which they must satisfy is, to some extent, arbitrary. The ones above were chosen because they are relatively simple and also, as will be shown in the next section, because they lead to accurate predictions for the rate of heat transfer in forced or in natural convection, where a comparison with the results of exact calculations is possible. The temperature profile given by Equation (13) is quite standard and has been chosen before for other problems, and the velocity profile in Equation (23) can be broken up into two parts. The expression $(2\eta - 2\eta^3 + \eta^4)$ has been used to represent the velocity profile in forced convection. On the other hand the term $(\delta^2\eta/6)(1-\eta)^3$ could be taken as a velocity profile in natural-convection heat transfer, although it appears that a somewhat different form has usually been selected (*9, 11*). Therefore the velocity profile in Equation (12) is the sum of two terms, the velocity profile for forced convection and that for natural-convection heat transfer. This is shown schematically in Figure 2.

If Equations (12) and (13) are then substituted into Equations (10) and (11) one obtains the following two ordinary differential equations for the variables δ and Δ:

$$\frac{d\delta^2}{dz} = \frac{4 \mp \frac{\delta^2}{3}\left(\frac{9\Delta}{4} - 1\right)}{0.11746 \mp 3.1750 \times 10^{-3}\,\delta^2 - 5.51145 \times 10^{-4}\,\delta^4} \quad (15)$$

and

$$\frac{d\Delta}{dz} = \frac{\frac{3}{Pr\Delta} - \left[H_1(\Delta) \pm \frac{\delta^2}{2}H_2(\Delta)\right]\frac{d\delta^2}{dz}}{2\delta^2\frac{dH_1(\Delta)}{d\Delta} \pm \frac{\delta^4}{3}\frac{dH_2(\Delta)}{d\Delta}} \quad (16)$$

where Δ, $H_1(\Delta)$, and $H_2(\Delta)$ are defined below.

$$\Delta = \frac{\delta_T}{\delta}$$

$$H_1(\Delta) = \left(\frac{\Delta^2}{5} - \frac{3\Delta^4}{70} + \frac{\Delta^5}{80}\right) \quad \text{for } \Delta \leq 1 \quad (17)$$

$$H_2(\Delta) = \left(\frac{\Delta^2}{10} - \frac{\Delta^3}{8} + \frac{9\Delta^4}{140} - \frac{\Delta^5}{80}\right) \quad \text{for } \Delta \leq 1$$

The initial conditions at $z = 0$ are

$$\delta = 0 \text{ and } \Delta H_1(\Delta) = \frac{0.088095}{Pr} \quad (18)$$

since $d\Delta/dz$ must remain finite. Its value at $z = 0$ can be obtained from Equation (16) by a limiting process, so that

$$\left(\frac{d\Delta}{dz}\right)_{z=0} = \mp\frac{5.67569H_2(\Delta) + 1.2528H_1(\Delta) - 2.1284\Delta H_1(\Delta)}{\frac{dH_1}{d\Delta} + \frac{0.0293649}{Pr\,\Delta^2}} \quad (18a)$$

If $\Delta > 1$, then $H_1(\Delta)$ and $H_2(\Delta)$ have different forms. However, it turns out that for Pr greater than about 0.6, which includes essentially most cases of interest except liquid metals, Δ remains less than 1 for all values of z. This is one of the reasons why Equations (12) and (13) were finally selected to represent the velocity and the temperature profiles respectively.

δ and Δ only intermediates in the determination of τ_0 and Nu. It is found that

$$\frac{6\tau_0}{\rho U_\infty^2}\sqrt{Re}\cdot\sqrt{\frac{Re^2}{Gr}} = \frac{12 \pm \delta^2}{\delta} = \Gamma_1 \quad (19)$$

and

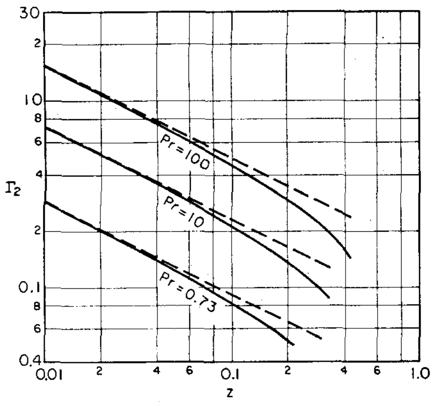

Fig. 6. Γ_2 **cooling vs.** z.

$$\frac{Nu}{\sqrt{Re}} \cdot \sqrt{\frac{Re^2}{Gr}} = \frac{1.5}{\delta\Delta} = \Gamma_2 \qquad (20)$$

from which τ_0 and Nu can be calculated once Equations (15) and (16) have been solved. It is interesting to note that for the heating problem τ_0 remains always positive, whereas for the cooling problem $\tau_0 = 0$ at $\delta^2 = 12$, which implies the existence of a separation point. This is expected from the equations of motion and the known result that, in laminar flow and in the absence of free convection, a separation point can exist when the term $U(dU/dx_1)$ in Equation (1) is negative (*10*). One can generalize therefore that (*a*) in the heating of upward flow over an arbitrary surface free convection will move the separation point away from the forward stagnation point, since the buoyancy term in Equations (1) is positive and (*b*), for the opposite reason, in the cooling of upward flow free convection will move the separation point toward the forward stagnation point. Thus it is easy to see that as $(Gr/Re^2) \to \infty$ Equations (5) and (6) reduce to the corresponding relations for natural-convection heat transfer for the heating problem only, for as $Gr/Re^2 \to \infty$ the separation point moves closer and closer to the forward stagnation point, and so, in general, the boundary-layer equations remain valid only over a smaller and smaller part of the surface.

HEATING PROBLEM FOR UPWARD FLOW OVER A VERTICAL FLAT PLATE

It is often advisable to compare, wherever possible, the results obtained by approximate methods with those derived from an exact solution of the mathematical equations. This is possible in the present problem for the two asymptotic regions only,

$z \to 0$ (natural convection is negligible,

and,

$z \to \infty$ (forced convection is negligible)

The comparison is shown below.

Case a: Asymptotic solution as $z \to 0$.

Since $\delta \to 0$ as $z \to 0$, it follows from Equations (15) and (16) that

$$\frac{d\delta^2}{dz} = 34.054$$

and

$$\Delta H_1(\Delta) = \frac{0.088095}{Pr} \qquad (21)$$

Therefore from Equation (19)

$$\frac{\tau_0}{\rho U_\infty^{\ 2}} \sqrt{Re} = 0.343$$

which compares very favorably with the value 0.332 obtained from an exact solution of the Blasius equation (*10*). On the other hand, the expression for the Nusselt number depends on Pr. It is found that for

$$Pr = 0.73, \quad \frac{Nu}{\sqrt{Re}} = 0.291,$$

compared with the exact expression

$$\frac{Nu}{\sqrt{Re}} = 0.297$$

for

$$Pr \to \infty, \quad \frac{Nu}{\sqrt{Re}} = 0.338\, Pr^{\frac{1}{3}},$$

compared with the exact result

$$\frac{Nu}{\sqrt{Re}} = 0.339\, Pr^{\frac{1}{3}} \qquad (5)$$

Case b. Asymptotic solution as $z \to \infty$.

Since $\delta \to \infty$ as $z \to \infty$, Equations (15) and (16) have the asymptotic forms

$$\frac{d\delta^2}{dz} = \frac{604.8}{\delta^2}\left(\frac{9\Delta}{4} - 1\right)$$

and

$$Pr = \frac{6}{604.8 H_2(\Delta)\left(\frac{9\Delta}{4} - 1\right)} \qquad (22)$$

respectively, which can also be derived by considering the free convection problem alone. One can easily show then that

$$\text{for } Pr = 0.73, \ \frac{Nu}{(Gr)^{\frac{1}{4}}} = 0.342$$

and

$$\text{for } Pr \to \infty, \ \frac{Nu}{(Gr)^{\frac{1}{4}}} = 0.480\, Pr^{\frac{1}{4}} \qquad (23)$$

These results are again in good agreement with those obtained by Schuh (*10*) and Ostrach (*8*) from the exact numerical solution of the boundary-layer equations for natural convection. Thus

$$\text{for } Pr = 0.73, \ \frac{Nu}{(Gr)^{\frac{1}{4}}} = 0.358$$

and

$$\text{for } Pr \to \infty, \ \frac{Nu}{(Gr)^{\frac{1}{4}}} = 0.503\, Pr^{\frac{1}{4}} \qquad (23a)$$

These comparisons between the momentum integral method and the exact solutions for the two cases $z \to 0$ and $z \to \infty$ are more than satisfactory. It is therefore not unreasonable to believe that Equations (19) and (20) will be sufficiently reliable for all values of z.

Equations (15) and (16) have been solved numerically for three values of the Prandtl number: $Pr = 0.73$, 10, and 100. The variables of primary interest Γ_1 and Γ_2 are shown plotted in Figures 3 and 5, together with their two asymptotes at $z = 0$ and $z = \infty$. These graphs show that the change from pure forced to pure free convection is gradual and that the mechanisms are nonadditive. It is seen also that the rule of McAdams (*6*) according to which, "If it is doubtful

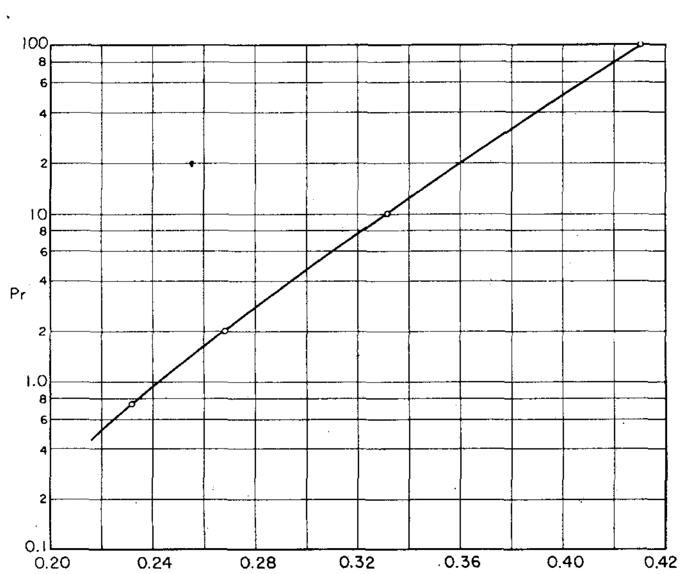

Fig. 7. z **separation vs. Prandtl number.**

whether forced or free convection flow applies, the heat transfer coefficient should be calculated using both the forced and the free convective relation and the larger one should be used," does not hold very well, especially for large Prandtl numbers.

The transition region, for which the two effects alone would be of the same order of magnitude, is most clearly seen in the plots of Γ_1 vs. z. It is interesting to note that according to Equations (15) and (16) Γ_1 is a minimum at $\delta^2 = 12$ and remains constant over a finite portion of the z axis, because the denominator in Equation (15) is zero at $\delta^2 = 12$ while the numerator is not. This unexpected result is due solely to the inherent inaccuracy of the momentum-integral method. It can be shown that, no matter which "reasonable" velocity profile is substituted in Equation (10), Γ_1 will have a minimum, approximately equal to 7.0, which will extend over a finite part of the z axis. It is believed, naturally, that if a more elaborate method for solving the boundary-layer equations is devised, one can obtain a smooth curve for Γ_1 vs. z, but it is improbable that the added accuracy thus arrived at would be significant. Therefore, in integrating Equations (15) and (16), one has to separate the z axis into three regions: (1) In the first region δ^2 increases from zero to 12, and Δ decreases from its initial value. (2) In the second region δ^2 remains constant at 12, but Δ increases, according to Equation (16), until it becomes equal to 8/9. (3) Finally Equations (15) and (16) are integrated for the third region, with initial conditions $\delta^2 = 12$ and $\Delta = 8/9$.

It is also clear from Figures 3 to 6 that it is very difficult to draw any specific conclusions which would allow one to predict, *a priori*, for what values of the parameter Gr/Re^2 forced or free convection alone would predominate. It is true that the influence of free convection is negligible for $z < 0.02$ for all Prandtl numbers; however, the value of z above which, for practical purposes, forced convection does not influence the shear stress or the rate of heat transfer depends on the Prandtl number. Thus for $Pr = 0.73$ this value of z is approximately 2; for $Pr = 100$ it is approximately 30. These inequalities are of course for the flat plate only. It is expected that somewhat different results would probably be reached for other surface geometries.

THE COOLING PROBLEM FOR UPWARD FLOW OVER A VERTICAL FLAT PLATE

The integration of Equations (15) and (16) is straightforward for the cooling problem, since they do not become singular for $0 < \delta^2 < 12$. When $\delta^2 = 12$, Γ_1 becomes equal to zero, which indicates the presence there of a separation point. Γ_1 and Γ_2 are plotted in Figures 4 and 6, and the separation point is shown as a function of the Prandtl number Pr in Figure 7. (This latter curve is based on four Prandtl numbers.) For most Prandtl numbers of interest the value of z at which separation occurs lies approximately between 0.2 and 0.5, but since it is known that the momentum-integral method cannot generally predict the separation point very accurately, the curve in Figure 7 should be considered only as an approximate one.

CONCLUSION

A method was presented for calculating the shear stress and the rate of heat transfer in external flows for combined laminar free and forced convection. The parameter Gr/Re^2 is of fundamental importance in such problems. Natural convection is negligible as $Gr/Re^2 \to 0$, and forced convection has little influence as $Gr/Re^2 \to \infty$. Numerical results were reported for the heating and cooling of upward flow past a vertical flat plate for three Prandtl numbers. It was found that the mechanisms of forced and free convection are nonadditive. The influence of free convection on the separation point was also examined, and it was shown that heating of upward flow stabilizes the boundary layer, whereas cooling hastens the appearance of the separation point.

The numerical results for the heating problem from a flat plate indicate, moreover, that the transition from a pure forced to pure free convection is gradual, especially at high Prandtl numbers. It is very difficult therefore to draw any specific conclusions concerning the values of the parameter Gr/Re^2 for which one of the above two mechanisms alone would predominate. It is observed that for the flat plate and for $z < 0.02$, free convection is negligible, whereas for $z > 100$ forced convection has little influence. It is expected however that somewhat different results would probably be reached for other surface geometries.

Sufficient experimental data do not exist to make a comparison with these theoretical deductions.

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NOTATION

x = coordinate measured along surface from forward stagnation point
y = coordinate normal to surface
u, v = velocity components in x and y directions respectively
ρ = density
ρ_∞ = density outside boundary layer
g = acceleration due to gravity
ν = kinematic viscosity
$\epsilon(x_1)$ = angle between normal to surface and direction of force of gravity (See Figure 1.)
T = temperature
T_w = temperature at surface
T_∞ = temperature of the fluid outside the boundary layer
k, c_p = thermal conductivity and specific heat respectively
$U(x)$ = the velocity outside the boundary layer
β = coefficient of thermal expansion
U_∞ = characteristic velocity
L = characteristic length
$U(x_1)$ = dimensionless potential flow distribution
$u_1, v_1, x_1, y_1, \theta, \beta_1, Re, Gr, Pr$ = defined by Equation (2)
$|T_w - T_\infty|$ = absolute value of the temperature difference $T_w - T_\infty$
Nu = Nusselt number = $-L(\partial\theta/\partial y)_{y=0}$
Nu_m = average Nusselt number
$\tau_0, \overline{\tau_0}$ = shear stress and average shear stress respectively
z = $Gr/Re^2 = \beta_1 x/U_\infty^2$ (For the flat plate one can set $L = x$ and therefore $Gr = \beta_1 x^3/\nu^2$ and $Re = U_\infty x/\nu$.)
$\Delta, H_1(\Delta), H_2(\Delta)$ are defined by Equation (17)
Γ_1, Γ_2 are defined by Equations (19) and (20) respectively

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